Eur. Phys. J. C 46, 771–776 (2006)
Digital Object Identifier (DOI) 10.1140/epjc/s2006-02527-5

THE EUROPEAN
PHYSICAL JOURNAL C

Pion condensation in electrically neutral cold matter with finite baryon density

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Received: 18 October 2005 /
Published online: 12 April 2006 –

Abstract. The possibility of the pion condensation phenomenon in cold and electrically neutral dense baryonic matter is investigated in β-equilibrium. For simplicity, the consideration is performed in the framework of a Nambu–Jona-Lasinio model with two quark flavors at zero current quark mass and for rather small values of the baryon chemical potential, where the diquark condensation might be ignored. Two sets of model parameters are used. For the first, the pion condensed phase with finite baryon density is realized. In this phase both electrons and the pion condensate take part in the neutralization of the quark electric charge. For the second set of model parameters, the pion condensation is impossible if the neutrality condition is imposed. The behavior of meson masses vs. quark chemical potential has been studied in electrically neutral matter.

PACS. 14.40.-n; 11.30.Qc; 12.39.-x; 21.65.+f

1 Introduction

According to a well-known point of view [1–3], pionic degrees of freedom and, especially, the pion condensation phenomenon might play a significant role in the description of different nuclear matter effects. At the present time it is widely believed that the dense baryonic matter that might exist inside compact star cores or observed in relativistic heavy ion collisions is no more than dense quark matter, obeying an isospin asymmetry. The physics of such a quark matter is adequately described in the framework of QCD with nonzero isospin chemical potential $\mu_{\rm I}$. Recently, it was shown that a nonzero pion condensate is generated in QCD if $\mu_{\rm I}$ is greater than the pion mass. This result was obtained in the framework of an effective chiral Lagrangian approach [4] as well as in QCD lattice calculations, performed at zero or small values of the baryon chemical potential $\mu_{\rm B}$ [5]. However, these two approaches are not applicable for the description of an isotopically asymmetric matter at moderate baryon density. To overcome the problem, it was proposed to study the QCD phase diagram on the basis of Nambu–Jona-Lasinio (NJL)-type models [6, 7] (see also the reviews [8, 9]), which contain quarks as microscopic degrees of freedom, in the presence of a baryon chemical potential $\mu_{\rm B}$ and an isospin $\mu_{\rm I}$ one. In this way the influence of $\mu_{\rm B}, \mu_{\rm I}$ on both the chiral symmetry restoration effect [10] and the formation of color superconducting (CSC) dense baryonic matter [11] was considered, but without taking into account the pion condensation phenomenon.

Recently, the pionic condensation effect was investigated in some NJL models at nonzero values of $\mu_{\rm B}$ and $\mu_{\rm I}$ [12–14]. In particular, it was shown in [14] that quark matter with finite isospin density might exist in two different phases. In the first the baryon density is zero and quarks are gapped, whereas in the second the baryon density is nonzero and quarks are gapless. Note that in [12–14] the chemical potentials $\mu_{\rm B}$, $\mu_{\rm I}$ are independent external parameters, so the results might be relevant to the physics of the heavy ion collision experiments only, and do not describe the real situation inside compact stars. The reason is that matter in the bulk of a compact star should be electrically neutral (at least, on average) as well as remain in β-equilibrium, i.e. all β-processes that include quarks and leptons should go with equal rates in both directions (as a rule, in this case $\mu_{\rm I}$ depends on $\mu_{\rm B}$).

In the present paper, we study in the framework of an NJL model, in contrast to [12–14], the possibility of the pion condensation phenomenon in electrically neutral matter with finite baryonic density at zero temperature. Moreover, matter in our consideration is required to be in β-equilibrium. This means that, apart from quark and meson degrees of freedom, it is necessary to take into account charged leptons (electrons only, for simplicity). Since both the pion condensate and electrons have a nonzero electric charge, it is clear that the positive charge of quark matter might be compensated in our case in several ways,

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depending on the competition between electrons and the pion condensate. We placed the charge neutrality condition only locally, i.e. we suppose that the ground state of matter is a uniform phase with zero electric charge density. One should also note that we do not take into consideration the CSC effects (see e.g. [15–19]), so our results are valid for not too large values of the baryon chemical potential, say, for $\mu_B < 1200\,\text{MeV}$[1].

The paper is organized as follows. In Sect. 2 the phase structure of the NJL model with zero current quark mass is investigated under the requirements of electrical neutrality and β-equilibrium for the two parameter sets: $G = 5.01\,\text{GeV}^{-2}$, $\Lambda = 0.65\,\text{GeV}$ (set 1) and $G = 6.82\,\text{GeV}^{-2}$, $\Lambda = 0.6\,\text{GeV}$ (set 2) (G is the model coupling constant, Λ is the three-dimensional cutoff parameter, used in loop integrations). It turns out that for the set 1 the neutral matter with pion condensate is allowed to exist at some values of $\mu \equiv \mu_B/3$, whereas for the set 2 it is forbidden. In Sect. 3 the mass behaviors of the scalar and pseudoscalar mesons are considered vs. μ for the parameter set 1.

2 The model and its phase structure

Our investigations are based on the NJL model with two quark flavors. The corresponding Lagrangian has the following form:

$$L_q = \bar{q}\gamma^\nu \mathrm{i}\partial_\nu q + G\left[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma^5\tau q)^2\right], \tag{1}$$

where τ_i $(i = 1, 2, 3)$ are Pauli matrices and, for simplicity, current quark masses are taken as zero. Clearly, the Lagrangian L_q is invariant under transformations of the color $\mathrm{SU_c}(3)$ and baryon $\mathrm{U_B}(1)$ groups as well as under the parity transformation P. In addition, this Lagrangian is symmetric with respect to the chiral $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ group (chiral transformations act on the flavor indices of quark fields only). In particular, it is invariant under the isotopic $\mathrm{SU(2)_I}$ group as well. Moreover, since $Q = I_3 + B/2$ (in the flavor space $I_3 = \tau_3/2$ is the generator of the third isospin component, $Q = \mathrm{diag}(2/3, -1/3)$ is the generator of the electric charge, and $B = \mathrm{diag}(1/3, 1/3)$ is the baryon charge generator), the electric charge is conserved too in the NJL model (1).

Due to the β-equilibrium requirement, we must incorporate electrons in our consideration. So, the full Lagrangian of the system looks like

$$\bar{L} = L_q + \bar{e}\gamma^\nu \mathrm{i}\partial_\nu e, \tag{2}$$

where e is the electron spinor field. (We suppose that electrons are free massless particles, for simplicity.) Clearly, the Lagrangian (2) is well suited for the description of different processes in the vacuum. To study the properties of matter with nonzero baryon as well as electric charges, we need to modify (2) as follows:

$$L = \bar{L} + \mu_B N_B + \mu_Q N_Q, \tag{3}$$

where N_B, N_Q are baryon- and electric-charge density operators, correspondingly, and μ_B, μ_Q are their chemical potentials[2]. Evidently,

$$N_B = \bar{q}B\gamma^0 q, \quad N_Q = \bar{q}Q\gamma^0 q - \bar{e}\gamma^0 e. \tag{4}$$

The μ_Q term in (3) spoils the vacuum chiral $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$ symmetry of the system. So, at $\mu_Q \neq 0$ the Lagrangian L is invariant only under the reduced $\mathrm{U_{I_3L}}(1) \times \mathrm{U_{I_3R}}(1)$ chiral symmetry group, i.e. the isotopic $\mathrm{SU(2)_I}$ symmetry between u and d quarks is absent in the medium. In this case the pion condensation phenomenon might occur. It means, without loss of generality, that the ground-state expectation value of the form $\langle\bar{q}\mathrm{i}\gamma^5\tau_1 q\rangle$ is nonzero, whereas $\langle\bar{q}\mathrm{i}\gamma^5\tau_{2,3}q\rangle = 0$ (clearly, parity is broken in the ground state of matter with nonzero pion condensate). Another characteristic of the ground state of dense matter is the chiral condensate, i.e. the quantity $\langle\bar{q}q\rangle$. When it is nonzero, the chiral symmetry is spontaneously broken down. In the present paper, in order to establish the phase structure of the neutral matter within the framework of the model (3), we restrict ourselves to the consideration of these two condensates only.

The competition between these two condensates is governed by the thermodynamic potential (TDP), which in the mean field approximation has the following form (it can be obtained with ease, using e.g. the technique of [19]):

$$\Omega(M, \Delta) = -\frac{\mu_Q^4}{12\pi^2} + \frac{M^2 + \Delta^2}{4G} - 3\sum_a \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,|E_a|, \tag{5}$$

where the first term on the right-hand side is the TDP of free massless electrons. The summation in (5) runs over all quasiparticles $(a = u, d, \bar{u}, \bar{d})$, where

$$\begin{aligned} E_u &= E_\Delta^- - \bar{\mu}, \qquad & E_{\bar{u}} &= E_\Delta^+ + \bar{\mu}, \\ E_d &= E_\Delta^+ - \bar{\mu}, \qquad & E_{\bar{d}} &= E_\Delta^- + \bar{\mu}, \end{aligned} \tag{6}$$

and $E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2}$, $E^\pm = E \pm \mu_Q/2$, $E = \sqrt{\mathbf{p}^2 + M^2}$, $\bar{\mu} = \mu_B/3 + \mu_Q/6$. The factor 3 in front of the summation symbol in (5) indicates the three-fold degeneracy of each quasiparticle in color. Moreover, in order to avoid usual ultraviolet divergences, the integration region in (5) is restricted by a cutoff Λ, i.e. $|\mathbf{p}| < \Lambda$. First of all, let us fix the model parameters as follows: $G = 5.01\,\text{GeV}^{-2}$, $\Lambda = 0.65\,\text{GeV}$ (set 1) (later, another parameter set will be discussed). The gap coordinates (M_0, Δ_0) of the global

[1] The properties of the electrically neutral and β-equilibrated CSC matter at finite baryon density were investigated in the framework of an NJL model in [16], but without taking into account the pion condensation.

[2] The Lagrangian (3) can be identically transformed in the following way: $L = \bar{L} + (\mu_B/3 + \mu_Q/6)\bar{q}\gamma^0 q + \mu_Q \bar{q} I_3 \gamma^0 q - \mu_Q \bar{e}\gamma^0 e$, where I_3 is presented after (1). It is clear from this relation that μ_Q is just the isospin chemical potential μ_I.

minimum point of the function $\Omega(M,\Delta)$ are connected with condensates in the following way:

$$M_0 = -2G\langle\bar{q}q\rangle\,, \quad \Delta_0 = -2G\mathrm{i}\langle\bar{q}\gamma^5\tau_1 q\rangle\,. \tag{7}$$

So, if Δ_0 is nonzero in the global minimum point (GMP), then the pion condensation phase is realized. Note that the quark gap M_0 is just the dynamical (constituent) quark mass. From (5) it is possible to obtain the gap equations

$$\begin{aligned} 0 &= \frac{\partial\Omega(M,\Delta)}{\partial M} \\ &\equiv \frac{M}{2G} - 6M\int\frac{\mathrm{d}^3p}{(2\pi)^3E}\left\{\frac{\theta(E_\Delta^+-\bar{\mu})E^+}{E_\Delta^+}\right. \\ &\qquad\left. + \frac{\theta(E_\Delta^--\bar{\mu})E^-}{E_\Delta^-}\right\}, \\ 0 &= \frac{\partial\Omega(M,\Delta)}{\partial\Delta} \\ &\equiv \frac{\Delta}{2G} - 6\Delta\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left\{\frac{\theta(E_\Delta^+-\bar{\mu})}{E_\Delta^+} + \frac{\theta(E_\Delta^--\bar{\mu})}{E_\Delta^-}\right\}. \end{aligned} \tag{8}$$

As was noted in the Introduction, we are going to impose the neutrality constraint locally, i.e. we search for the ground state of the system, in which the electric charge density $n_Q \equiv -\partial\Omega/\partial\mu_Q$ turns into zero. In other words, we study the GMP of the function $\Omega(M,\Delta)$ under the constraint

$$\begin{aligned} 0 = n_Q &\equiv \frac{\mu_Q^3}{3\pi^2} + \int\frac{\mathrm{d}^3p}{(2\pi)^3} \\ &\times\left\{\theta(\bar{\mu}-E_\Delta^+) + \theta(\bar{\mu}-E_\Delta^-)\right. \\ &\qquad\left. + 3\theta(E_\Delta^+-\bar{\mu})\frac{E^+}{E_\Delta^+} - 3\theta(E_\Delta^--\bar{\mu})\frac{E^-}{E_\Delta^-}\right\}. \end{aligned} \tag{9}$$

It is easily seen from the gap equations (8) that at $\mu_Q \neq 0$ the global minimum point of the TDP (5) might take only one of the following three forms in the (M,Δ)-space: (i) $(0,0)$, (ii) $(M_0,0)$, and (iii) $(0,\Delta_0)$. (In the ground state, corresponding to the GMP of the form (i), both the chiral and pion condensates are zero. The solution of the type (ii) corresponds to the matter phase in which only the chiral condensate is generated. Finally, in the GMP of the form (iii) the chiral condensate is zero, but the pion condensate is nonzero[3].) If the neutrality requirement (9) is not taken into account, then the quantities M_0 and Δ_0 are functions of the chemical potentials μ_Q and μ_B, which are independent quantities. In this case the model is relevant for the description of matter with nonzero electric charge density.

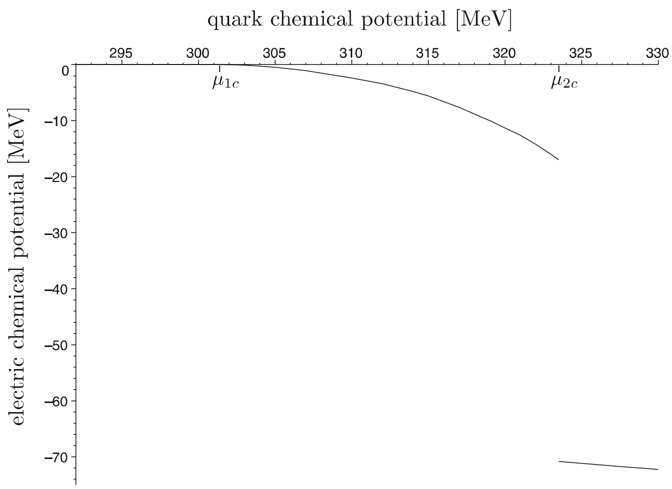

Fig. 1. The behavior of the electric chemical potential μ_Q vs. quark chemical potential $\mu = \mu_B/3$ in the electrically neutral matter for set 1 of NJL model parameters. Here $\mu_{1c} \approx 301$ MeV, $\mu_{2c} \approx 323.5$ MeV

However, since we are going to study the electrically (locally) neutral matter, it is necessary to perform the joint consideration of the neutrality constraint (9) and the gap equations (8). In this case both the gaps, i.e. the GMP coordinates, and the electric chemical potential μ_Q are functions of the baryon chemical potential μ_B only (below, the analysis is performed in terms of the quark chemical potential $\mu \equiv \mu_B/3$). Numerical investigations of the equations (8) and (9) show the following results on the phase structure of the electrically neutral matter for the parameter set 1.

At sufficiently small quark chemical potential $\mu < \mu_{1c} \approx 301$ MeV the quantity μ_Q is equal to zero (see Fig. 1). In this case the ground state of the system corresponds to a chirally noninvariant phase. In terms of TDP it means that at $\mu < \mu_{1c}$ the GMP has the form (ii) with $M_0 \approx 301$ MeV. In this phase both the baryon and the isospin densities are zero, so there is no need to neutralize the quark electric charge. Hence, it is not surprising why $\mu_Q = 0$ at rather small values of μ.

At $\mu_{1c} < \mu < \mu_{2c} \approx 323.5$ MeV the GMP has the form (iii), i.e. in this case the pion condensed phase is realized. The pion condensate Δ_0 vs. μ is depicted in Fig. 2. For this phase of neutral matter μ_Q is a nonzero negative quantity (see Fig. 1). Nevertheless, the numerical study shows that μ_Q is a sufficiently small quantity, so the relation $\Delta_0 < \bar{\mu}$ is fulfilled. As a result, one can see that both u- and d-quasiparticles are gapless in this phase, i.e. the

[3] If $\mu_Q = 0$, then the TDP (5) depends effectively on the single variable $\rho \equiv \sqrt{M^2+\Delta^2}$. So, at sufficiently small values of μ_B the global minimum of the function $\Omega(M,\Delta)$ is achieved at all points of some circle in the (M,Δ)-space. Formally, in this case there is a freedom for selecting the GMP. However, since at zero isospin chemical potential, i.e. at $\mu_Q = 0$, parity is a conserved quantity in the strongly interacting physics, we suppose that in this case the pion condensate is zero, but the chiral one is nonzero (at rather small values of μ_B), i.e. the global minimum is placed by hand in the point of the form (ii), and the chirally noninvariant phase is realized (the details of the phase structure investigation of the NJL model (3) with zero current quark mass and at $\mu_Q = 0$ are presented e.g. in [20]).

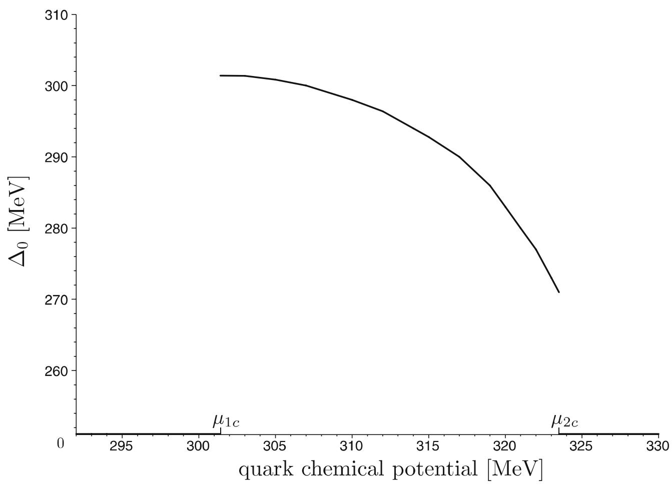

Fig. 2. The pion condensate Δ_0 vs. $\mu = \mu_B/3$ in the electrically neutral matter for the parameter set 1. Here $\mu_{1c} \approx 301$ MeV, $\mu_{2c} \approx 323.5$ MeV

quantities E_u and E_d from (6) with $\Delta = \Delta_0$ and $M = 0$ become zero at some energy values, so there is no energy cost for creating these quasiparticles. This fact means that at $\mu_{1c} < \mu < \mu_{2c}$ we have the gapless pion condensed phase (GPC), which has a nonzero baryon charge (see also [14]). In the GPC phase the electric charge of quarks is neutralized both by the electric charge of the electron gas and the charge of the pion condensate.

Finally, at larger values of the quark chemical potential, i.e. at $\mu_{2c} < \mu$, the normal dense baryonic phase is arranged, in which both condensates are zero. Hence, in this phase only the electron gas takes part in the neutralization of the quark electric charge. Therefore, the absolute values of μ_Q at $\mu_{2c} < \mu$ are greater than in the GPC phase (see Fig. 1).

Now, let us consider the phase structure of the electrically neutral matter for another set of model parameters (set 2): $G = 6.82\,\text{GeV}^{-2}$, $\Lambda = 0.6$ GeV (in this case the dynamical quark mass is approximately 400 MeV in the vacuum, i.e. at μ_Q, $\mu_B = 0$). Numerical analysis shows that for set 2 the phase structure of the model (3) differs qualitatively from the set 1 case. Indeed, here at the point $\mu_c \approx 386.2$ MeV we have at once the phase transition from the chirally noninvariant phase, which is at $\mu < \mu_c$, to the normal dense baryonic phase, at $\mu_c < \mu$, with zero pion – as well as chiral condensates. It turns out that under the neutrality constraint the pion condensation is prohibited in the NJL model with set 2 parameters at zero temperature[4]. In contrast, without this constraint the pion condensation is allowed to exist in the set 2 NJL model (see [12, 14]).

[4] This is an important point. Indeed, it was shown in [16, 17] that in electrically neutral matter with rather large baryon density the temperature induces a diquark condensation for some range of model parameters, thus tending to the color superconductivity. In a similar way, we suppose that at low baryon density the pion condensation can appear in the neutral matter at some temperature interval. However, it is a subject of special consideration.

3 Meson masses in electrically neutral matter

In the present section the masses of mesons are investigated in the electrically neutral matter. We will follow the method used in [21] for studying the particle masses in the color superconducting quark matter. To begin with, let us introduce auxiliary bosonic fields

$$\sigma(x) = -2G(\bar{q}q)\,, \quad \pi_a(x) = -2G(\bar{q}\mathrm{i}\gamma^5\tau_a q)\,, \tag{10}$$

where $a = 1, 2, 3$. In the following we will ignore the influence of electrons on the in-medium meson masses. In terms of $\sigma(x)$ and $\pi_a(x)$ the Lagrangian (3) (with omitted electron part) can be reduced to the form

$$L = \bar{q}\left[\gamma^\nu \mathrm{i}\partial_\nu + \bar{\mu}\gamma^0 + \frac{\mu_Q}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\right] \times q - \frac{1}{4G}\left[\sigma\sigma + \pi_a\pi_a\right] \tag{11}$$

(the quantity $\bar{\mu}$ is defined after (6)). Starting from (11), it is possible to integrate out the quark fields and obtain the effective action of the system in the one-quark loop approximation:

$$\mathcal{S}_{\text{eff}}(\sigma, \pi_a) = -\int \mathrm{d}^4x \left[\frac{\sigma^2 + \pi_a^2}{4G}\right] - \mathrm{i}\mathrm{Tr}_{sfcx} \ln D\,, \tag{12}$$

where

$$D = \gamma^\nu \mathrm{i}\partial_\nu + \bar{\mu}\gamma^0 + \frac{\mu_Q}{2}\tau_3\gamma^0 - \sigma - \mathrm{i}\gamma^5\pi_a\tau_a\,. \tag{13}$$

The Tr operation in (12) stands for calculating the trace in spinor (s), flavor (f), color (c) as well as four-dimensional coordinate (x) spaces, correspondingly.

It is clear from (7) and (10) that the coordinates (M_0, Δ_0) of the global minimum point of the TDP are just the ground-state expectation values of the σ and π_1 fields, i.e. $M_0 \equiv \langle\sigma(x)\rangle$, $\Delta_0 \equiv \langle\pi_1(x)\rangle$.

Let us make the following field shifts in (12): $\sigma(x) \to M_0 + \sigma(x)$, $\pi_1(x) \to \Delta_0 + \pi_1(x)$, and thereafter expand the effective action up to second order in the meson fields. Differentiating twice the obtained expression with respect to meson fields, it is then possible to obtain the one-particle irreducible (1PI) Green's functions Γ_{XY} of the mesons $(X, Y = \sigma, \pi_1, \pi_2, \pi_3)$. (In the present paper we omit these cumbersome calculations, referring to the similar meson mass calculations in [21].) The results are the following.

First, let us consider the masses of the σ- as well as π_3-mesons. (Note that $\pi_0 \equiv \pi_3$.) It turns out that both σ- and π_0-mesons are not mixed with other particles. Moreover, in the momentum space representation and at zero three-momentum, $\mathbf{p} = 0$, we have in the neutral gapless pion condensed phase of matter ($\mu_{1c} < \mu < \mu_{2c}$):

$$\Gamma_{\sigma\sigma}(p_0) = \Gamma_{\pi_0\pi_0}(p_0)$$

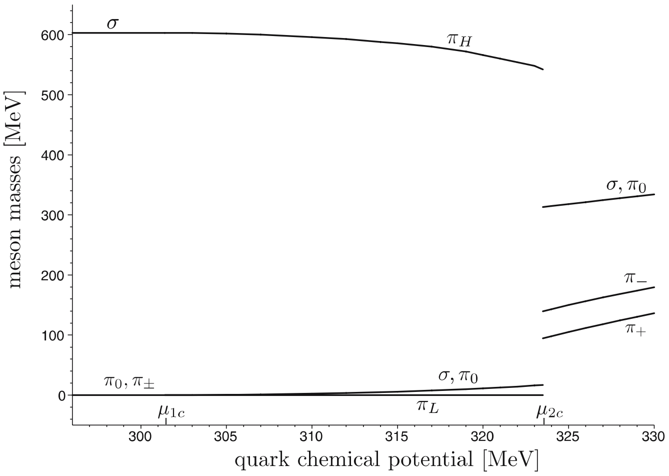

Fig. 3. Scalar and pseudoscalar meson masses vs. $\mu = \mu_B/3$ in the electrically neutral matter for the parameter set 1. Here $\mu_{1c} \approx 301$ MeV, $\mu_{2c} \approx 323.5$ MeV

$$
\begin{aligned}
&= \frac{1}{2G} + 6\int \frac{\mathrm{d}^3q}{(2\pi)^3} \\
&\times \left\{ \frac{\theta(\varepsilon^+_\Delta - \bar{\mu})}{\varepsilon^+_\Delta} \left[\frac{\varepsilon^+_\Delta p_0 - \mu_Q \varepsilon^+}{(\varepsilon^+_\Delta - p_0)^2 - (\varepsilon^-_\Delta)^2} \right.\right. \\
&\qquad \left. - \frac{\varepsilon^+_\Delta p_0 + \mu_Q \varepsilon^+}{(\varepsilon^+_\Delta + p_0)^2 - (\varepsilon^-_\Delta)^2} \right] \\
&+ \frac{\theta(\varepsilon^-_\Delta - \bar{\mu})}{\varepsilon^-_\Delta} \left[\frac{\varepsilon^-_\Delta p_0 + \mu_Q \varepsilon^-}{(\varepsilon^-_\Delta - p_0)^2 - (\varepsilon^+_\Delta)^2} \right. \\
&\qquad \left.\left. + \frac{\mu_Q \varepsilon^- - \varepsilon^-_\Delta p_0}{(\varepsilon^-_\Delta + p_0)^2 - (\varepsilon^+_\Delta)^2} \right] \right\},
\end{aligned}
\tag{14}
$$

where $\varepsilon^\pm_\Delta = \sqrt{(\varepsilon^\pm)^2 + \Delta_0^2}$, $\varepsilon^\pm = |q| \pm \mu_Q/2$. Recall that $\bar{\mu} = \mu + \mu_Q/6$. Eliminating in (14) the coupling constant G with the help of the Δ_0-gap equations (8), we obtain in the GPC phase

$$\Gamma_{\sigma\sigma}(p_0) = \Gamma_{\pi_0\pi_0}(p_0) \sim p_0^2 - \mu_Q^2\,. \tag{15}$$

Since the zero of a 1PI function in the p_0^2-plane defines the mass squared of a particle, it is evident from (15) that in the GPC phase $M_{\pi_0} = M_\sigma = |\mu_Q|$ (see Fig. 3, where M_{σ,π_0} are depicted, or Fig. 1 for μ_Q).

The expressions for the 1PI functions of the σ- and π_0-mesons in the normal dense quark matter, i.e. at $\mu_{2c} < \mu$, follow from (14) at $\Delta_0 = 0$. The zeros of the Green's functions $\Gamma_{\sigma\sigma}(p_0)$ and $\Gamma_{\pi_0\pi_0}(p_0)$ in the p_0^2-plane were studied numerically in this phase as well. The corresponding mass behaviors vs. μ are depicted in Fig. 3 at $\mu_{2c} < \mu$.

In contrast to the σ, π_0-sector, in the sector of π_1 and π_2 fields the 1PI Green's functions (at $\mathbf{p} = 0$) form a nontrivial matrix $\Gamma^{\mathrm{GPC}}(p_0)$ in the GPC phase, i.e. there is a mixing between π_1 and π_2[5]. Its matrix elements are

$$
\begin{aligned}
&\Gamma^{\mathrm{GPC}}_{\pi_1\pi_1}(p_0) = 6(p_0^2 - 4\Delta_0^2)A(p_0^2)\,, \\
&\Gamma^{\mathrm{GPC}}_{\pi_2\pi_2}(p_0) = 6p_0^2 A(p_0^2)\,, \\
&\Gamma^{\mathrm{GPC}}_{\pi_2\pi_1}(p_0) = \Gamma^{\mathrm{GPC}}_{\pi_1\pi_2}(-p_0) = 12\mathrm{i}p_0 B(p_0^2)\,,
\end{aligned}
\tag{16}
$$

where Δ_0 is presented in Fig. 2 and

$$
\begin{aligned}
A(p_0^2) &= \int \frac{\mathrm{d}^3q}{(2\pi)^3} \left\{ \frac{\theta(\varepsilon^+_\Delta - \bar{\mu})}{\varepsilon^+_\Delta[p_0^2 - 4(\varepsilon^+_\Delta)^2]} + \frac{\theta(\varepsilon^-_\Delta - \bar{\mu})}{\varepsilon^-_\Delta[p_0^2 - 4(\varepsilon^-_\Delta)^2]} \right\}, \\
B(p_0^2) &= \int \frac{\mathrm{d}^3q}{(2\pi)^3} \left\{ \frac{\varepsilon^+\theta(\varepsilon^+_\Delta - \bar{\mu})}{\varepsilon^+_\Delta[p_0^2 - 4(\varepsilon^+_\Delta)^2]} - \frac{\varepsilon^-\theta(\varepsilon^-_\Delta - \bar{\mu})}{\varepsilon^-_\Delta[p_0^2 - 4(\varepsilon^-_\Delta)^2]} \right\}
\end{aligned}
\tag{17}
$$

(see also the notation after (14)). Note that in the GPC phase $\Delta_0 < \bar{\mu}$ and, in addition, $\mu_Q < 0$. Therefore, the minimal value of the quantity ε^+_Δ in the integrands of (17) is $\bar{\mu}$. As a result, we see that $A(p_0^2)$ and $B(p_0^2)$ are analytical functions in the whole p_0^2-plane, except for the cut which is at $p_0^2 > 4\bar{\mu}^2$. Since there is a mixing between π_1 and π_2 fields, the masses of meson modes in this sector are defined by the zeros of $\det(\Gamma^{\mathrm{GPC}}(p_0))$ in the p_0^2-plane, i.e. by the equation

$$\det(\Gamma^{\mathrm{GPC}}(p_0)) = 36p_0^2\{(p_0^2 - 4\Delta_0^2)A^2(p_0^2) - 4B^2(p_0^2)\} = 0\,. \tag{18}$$

The evident solution of this equation is $p_0^2 = 0$. It corresponds to a massless meson mode, specified by π_L (see Fig. 3), that is actually the Nambu–Goldstone boson. (The appearance of such a mode in the GPC phase is justified by the spontaneous breaking of the initial $\mathrm{U}_{I_3\mathrm{L}}(1) \times \mathrm{U}_{I_3\mathrm{R}}(1)$ chiral symmetry down to the Abelian subgroup.) The nontrivial solution of (18) is the zero of the expression in the braces. It corresponds to a massive meson mode, denoted by π_H (see Fig. 3). (Clearly, its mass, M_{π_H}, lies in the interval $2\Delta_0 < M_{\pi_H} < 2\bar{\mu}$.)

In the dense normal quark matter (NQM) phase, i.e. at $\mu_{2c} < \mu$, it is convenient to use the charged fields $\pi_\pm(x) = (\pi_1(x) \pm \mathrm{i}\pi_2(x))/\sqrt{2}$. Then, in the NQM phase the matrix $\Gamma^{\mathrm{NQM}}(p_0)$ of 1PI Green's functions of the $\pi_\pm$-mesons looks like

$$
\begin{aligned}
\Gamma^{\mathrm{NQM}}_{\pi_+\pi_-}(p_0) &= \Gamma^{\mathrm{NQM}}_{\pi_-\pi_+}(-p_0) \\
&= \frac{1}{2G} - \frac{3}{\pi^2}\left\{ \Lambda^2 - \frac{3\mu_Q^2}{4} - \frac{p_0\mu_Q}{2} - \left(\mu + \frac{\mu_Q}{6}\right)^2 \right. \\
&\qquad \left. + \frac{(\mu_Q + p_0)^2}{4} \ln\left[\frac{4\Lambda^2 - (\mu_Q + p_0)^2}{(2\mu + \mu_Q/3)^2 - p_0^2}\right] \right\}, \\
\Gamma^{\mathrm{NQM}}_{\pi_+\pi_+}(p_0) &= \Gamma^{\mathrm{NQM}}_{\pi_-\pi_-}(p_0) = 0\,.
\end{aligned}
\tag{19}
$$

The numerical investigation of the zeros of the quantity $\det(\Gamma^{\mathrm{GPC}}(p_0))$ shows the presence of two pionic massive

[5] Note that at nonzero current quark mass there is actually a mixing between σ-, π_1-, and π_2 fields in the pion condensed phase.

modes, which might be identified in the NQM phase with $\pi_{\pm}$-mesons (see Fig. 3). Evidently, in this phase the mass splitting between $\pi_{\pm}$-mesons is due to the isospin asymmetry that is generated by nonzero μ_Q.

Finally, in Fig. 3 the masses of mesons in the chirally noninvariant phase, i.e. at $\mu < \mu_{1c}$, are presented as well. In this phase $\mu_Q \equiv 0$, and only the chiral gap $M_0 \neq 0$. It is well known that in this case $M_\sigma = 2M_0 \approx 602\,\mathrm{MeV}$, and the three π-mesons are massless Nambu–Goldstone bosons.

4 Summary

In the present paper we have studied the properties of electrically neutral and β-equilibrated cold matter with finite baryonic density. The problem is inspired by the physics of compact stars. For simplicity, the consideration was done in the framework of a NJL model with zero current quark mass. We have found that for the set 1 of model parameters (see the end of the Introduction) there are three different phases, including the one with a pion condensate, of neutral matter. In contrast, for the parameter set 2 the pion condensation in the neutral matter is forbidden. Moreover, we have studied the behavior of meson masses vs. quark chemical potential in the case of the parameter set 1 (see Fig. 3). Since the electric neutrality of the system is realized together with an isospin asymmetry between quarks, it turns out that the masses of π-mesons are split at nonzero baryon density.

Acknowledgements. This work was supported in part by DFG Project No. 436 RUS 113/477/0-2 and RFBR Grant No. 05-02-16699.

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